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INVASION ZONES IN LATERAL DRILLING

V. V. Shelukhin[1] **and I. N. Yeltsov**[2]

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A mathematical model for mud filtrate invasion in lateral drilling is proposed. The main assumption is that the difference in density between the invading and formation fluids is insignificant. Gravitational asymmetry of the invasion front is determined, and it is established that with time one of its points becomes abnormal and the entire invasion zone loses convexity. The main reason for the asymmetry is the density difference. If a lighter drilling mud is injected, the front "floats up"; if the mud is heavier, the front "floats down." The abnormal point of the front appears below or above the borehole, depending on the drilling mud weight. In the case where the mud is lighter than the formation fluid, the point of the front directly under the center of the borehole has the special property that with time it is less advanced downward than the neighboring left and right points of the front if the advance is reckoned from the horizontal axis through the center of the borehole. This property is the most pronounced for a small pressure difference between the borehole and formation equal to a certain critical value: under such conditions, the indicated point of the front does not move at all. For large pressure differences, the frontal advance is nearly equal in all directions.

Key words: *filtration, horizontal hole, displacement front.*

Introduction. During formation drilling, the mud invading the borehole zone, as a rule, has different physical properties than the formation fluids. As a result, the characteristics of the invasion zone differ from those of the uninvaded zone of the formation. In the invaded zone, nonuniform distributions of the electric resistance, oil saturation, salt content, and other important characteristics are observed. For nearly all modern geophysical tools of borehole research, the invaded zone is a hindering object. To correctly determine the formation characteristics, it is necessary to know the properties of the invaded zone. For straight-hole drilling using clay drilling muds, a combined geophysical and hydrodynamic model of the borehole zone was developed. A method of integrated interpretation of electromagnetic logs and technological drilling parameters based on hydrodynamic modeling of invasion was proposed [1].

However, in the last decade, horizontal drilling has found numerous applications. Apart from conventional drilling muds, oil-based muds have been used, and the excess of the borehole pressure over the formation pressure can be insignificant or even equal to zero. In this case, mud filtrate invasion into the formation can be substantially asymmetric because of gravity. Physical experiments and computer simulations have shown that such asymmetry is indeed the case [2]. Visual observations and measurements using industrial radars in experiments with water-saturated sand [3] have revealed an unusual anomaly of the brine invasion front (Fig. 1). The reasons for this are still unclear, and, require further research, including theoretical one.

The present paper deals with mud invasion during lateral drilling. Emphasis is on the effect of the pressure difference between the borehole and formation on the geometry of the invasion front. Therefore, we do not consider the case of multiphase filtration, ignore the anisotropy of the formation, and assume that the densities of the mud and formation fluid differ insignificantly.

[1]Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; shelukhin@hydro.nsc.ru. [2]Institute of Geophysics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; yeltsov@uiggm.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 45, No. 6, pp. 72–82, November–December, 2004. Original article submitted March 12, 2004; revision submitted April 8, 2004.

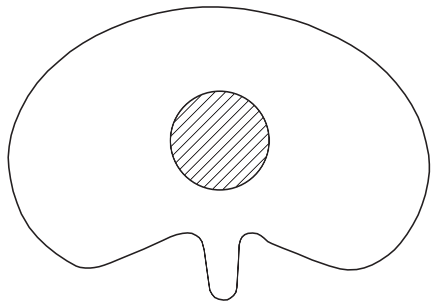


Fig. 1

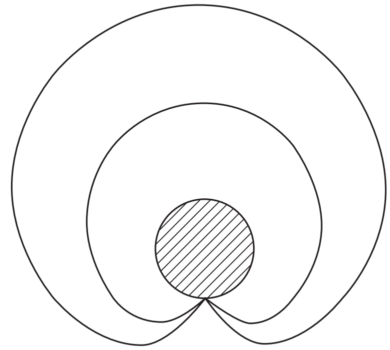


Fig. 2

It is shown that even under these assumptions, the invaded zone loses convexity with time. In this case, the point of the front that moves away from the center of the borehole more slowly than the others ultimately appears less advanced (compared to the neighboring points on the left and right) not only from the center of the borehole but also from the horizontal axis through the center of the borehole. We note that this result was obtained mainly by means of theoretical analysis. In addition, there is a certain critical pressure difference between the borehole and the formation δ_p^*. Thus, if the formation fluid weight exceeds the mud weight and $\delta_p = \delta_p^*$, the front at the lower point of the borehole does not move, i.e., there is no mud filtrate invasion through this point of the borehole (Fig. 2). For $\delta_p > \delta_p^*$, invasion through the lower point occurs but at a slower rate than through the neighboring points of the borehole. Conversely, if the formation fluid weight is lower than the drilling mud weight, then for $\delta_p = \delta_p^*$, the front does not move at the upper point of the borehole; for $\delta_p > \delta_p^*$ invasion through this point occurs but at a slower rate than through the neighboring points of the borehole.

1. Mathematical Model. The mathematical model proposed below describes the simple situation where the invading and formation fluids have identical viscosities and the mud density ρ_w differs from the density of the formation fluid ρ_f only slightly. In this case, the fluid invading the formation loses clay particles on the borehole wall, overcomes the additional resistance of the filtrate cake, and, when entering the formation, has the same density as the formation fluid [4].

The flow pattern is assumed identical in each cross-sectional plane with the coordinates x_1 and x_2. The cross section of the borehole is a circle of radius ε with center at the coordinate origin. The x_2 axis is upright, and the x_1 axis is directed to the right parallel to the plane tangent to the ground. We consider the flow in the region separated from the center of the borehole by a distance not larger than R; therefore cylindrical axes (r, φ) are used below, and the angle φ is reckoned from the x_1 axis upward.

Since the density of the formation fluid is constant, the salt distribution in the porous medium is described by the following equations (see [5]):

$$\Phi c_t + \operatorname{div}(\boldsymbol{q}c) = 0, \qquad \operatorname{div}\boldsymbol{q} = 0, \qquad \boldsymbol{q} = -k(\nabla p + \gamma_f \nabla z), \qquad \gamma_f = \rho_f g. \tag{1.1}$$

The flow is considered in a ring $\varepsilon < r < R$. The first equation is the transport equation, the second equation is the incompressibility condition, and the third is the Darcy filtration law. The following notation is adopted: Φ is the porosity, c is the mass concentration of the salt, $\boldsymbol{q}$ is the filtration velocity vector, p is the pressure, k is the filtration coefficient, g is the acceleration of gravity, γ_f is the formation fluid weight, and z is the depth function ($z = x_2$).

Away from the borehole, the pressure is distributed under the hydrostatic law, and, hence, the following boundary condition is satisfied:

$$r = R: \qquad H \equiv p + \gamma_f z = p_f = \text{const}. \tag{1.2}$$

The equality

$$r = \varepsilon: \qquad \boldsymbol{q} \cdot \boldsymbol{n} = -\beta[p] \tag{1.3}$$

is the boundary condition on the borehole wall ($\boldsymbol{n}$ is the outward normal vector to the circumference $r = \varepsilon$). Here β^{-1} is the resistance coefficient and

$$[p] = \lim_{\delta\to 0}\Big(p(1+\delta,\varphi) - p(1-\delta,\varphi)\Big)$$

is the pressure jump. Condition (1.3) implies that the invasion velocity of the mud liquids depends linearly on the pressure jump. As follows from (1.2), p_f is the formation pressure at the level of the center of the borehole at a distance R from it.

Another assumption is that the head $H_w \equiv p + \rho_w g z$ on the borehole wall is identical for all particles of the mud. This, in particular, is the case if the mud circulating in the annulus space is described by the Euler hydrodynamic equations of irrotational flow. In this case, the Bernoulli integral holds, i.e.,

$$\rho_w \boldsymbol{v}^2/2 + p + \rho_w g z = \text{const.}$$

Therefore, the head is identical for all particles having the same velocity $|\boldsymbol{v}|$. Thus, on the borehole wall, the head is constant:

$$r = \varepsilon: \qquad \lim_{\delta\to 0} p(1-\delta,\varphi) + \gamma_w z = p_w \equiv \text{const}, \quad \gamma_w = \rho_w g.$$

Hence, equality (1.3) can be written as

$$r = \varepsilon: \qquad H_r - \beta_1 H + \beta_1(\delta_\gamma r\sin\varphi + p_w) = 0, \quad \beta_1 = \beta/k, \quad \delta_\gamma = \gamma_f - \gamma_w. \tag{1.4}$$

System (1.1) should be supplemented by the boundary and initial conditions for the salt concentration:

$$c\Big|_{r=\varepsilon} = c_1, \qquad c\Big|_{t=0} = c_0. \tag{1.5}$$

We note that the first of conditions (1.5) for c is meaningful only for those points of the circle $r = \varepsilon$ at which the velocity $\boldsymbol{q}$ is directed into the formation, i.e., the following inequality should be satisfied:

$$\frac{\partial H}{\partial n} \leqslant 0 \qquad \text{at} \quad r = \varepsilon. \tag{1.6}$$

Below, this condition will be formulated in terms of the parameter δ_p.

2. Equations for the Invasion Front. The head H does not depend on time and is found as the solution of the following boundary-value problem in polar variables:

$$\Delta H \equiv r(rH_r)_r + H_{\varphi\varphi} = 0, \quad H\Big|_{r=R} = p_f, \quad H_r - \beta_1 H + \beta_1(\delta_\gamma r\sin\varphi + p_w)\Big|_{r=\varepsilon} = 0.$$

Using the method of separation of variables, we obtain

$$H = p_f - b_3\ln(r/R) + (-b_1 r + b_2 r^{-1})\sin\varphi, \tag{2.1}$$

where

$$b_1 = \frac{\beta_1\varepsilon\delta_\gamma}{(1-\beta_1\varepsilon) + (1+\beta_1\varepsilon)(R/\varepsilon)^2}, \quad b_2 = b_1 R^2, \quad b_3 = \frac{\delta_p}{(\beta_1\varepsilon)^{-1} + \ln(R/\varepsilon)}, \quad \delta_p \equiv p_w - p_f.$$

This solution implies that the flow pattern is symmetric about the vertical axis. We find the stream function ψ from the conditions $\psi_{x_1} = kH_{x_2}$ and $\psi_{x_2} = -kH_{x_1}$ using the curvilinear integral

$$\psi(x_1,x_2) = \int\limits_{(x_1^0,x_2^0)}^{(x_1,x_2)} -kH_{x_1}\,dx_2 + kH_{x_2}\,dx_1 = \int\limits_{L_1}\ldots + \int\limits_{L_2}\cdots, \qquad (x_1,x_2) = (r\cos\varphi, r\sin\varphi).$$

Here $(x_1^0, x_2^0) = (\varepsilon, 0)$. The segment L_1 connects the point (x_1^0, x_2^0) and the point $(x_1^*, x_2^*) = (r, 0)$. The segment L_1 is defined in the parametric form:

$$L_1: \qquad x_1 = \varepsilon + \mu(r-\varepsilon), \quad x_2 = 0, \quad 0 \leqslant \mu \leqslant 1.$$

The curvilinear segment L_2 is the part of the circumference defined in the parametric form

$$L_2: \qquad x_1 = r\cos(\mu\varphi), \quad x_2 = r\sin(\mu\varphi), \quad \tan\varphi = \frac{x_1}{x_2}, \quad 0 \leqslant \mu \leqslant 1.$$

Each integral $\int\limits_{L_i}$ is calculated by the formula

$$\int\limits_{L_i} \ldots = \int\limits_0^1 -kH_{x_1}(x_1(\mu), x_2(\mu))x_2'(\mu) + kH_{x_2}(x_1(\mu), x_2(\mu))x_1'(\mu)\, d\mu,$$

where $x_1 = x_1(\mu)$, $x_2 = x_2(\mu)$ is the parametric specification of the segment L_i. Finally, we obtain the following representation for the stream function ψ in polar coordinates:

$$\psi(r, \varphi) = kb_3\varphi - k(b_1 r + b_2/r)\cos\varphi + \text{const}.$$

Now, the trajectory of each liquid particle that issues from the point of the borehole (ε, φ_0) is specified by the equality $\psi(r, \varphi) = \psi(\varepsilon, \varphi_0)$.

The transport equation in polar coordinates is written as

$$c_t - \lambda(c_r H_r + r^{-2} c_\varphi H_\varphi) = 0 \qquad (\lambda = k/\Phi)$$

or

$$c_t + \lambda[b_3 r^{-1} + (b_1 + b_2 r^{-2})\sin\varphi]c_r - \lambda\cos\varphi(-b_1 r^{-1} + b_2 r^{-3})c_\varphi = 0.$$

It is known that such equations are solved by constructing characteristics along which the value of the solution remains unchanged. In the space of the variables r, φ, and t, the characteristics are specified by the equations

$$\begin{aligned} \frac{dr}{dt} &= \lambda[b_3 r^{-1} + (b_1 + b_2 r^{-2})\sin\varphi], \qquad r(0) = \varepsilon, \\ \frac{d\varphi}{dt} &= \lambda[b_1 r^{-1} - b_2 r^{-3}]\cos\varphi, \qquad \varphi(0) = \varphi_0 \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big]. \end{aligned} \tag{2.2}$$

To construct the invasion front, it suffices to find the solution of system (2.2):

$$r = r(t, \varphi_0), \qquad \varphi = \varphi(t, \varphi_0), \qquad \varphi_0 \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big]. \tag{2.3}$$

Equalities (2.3) define a parametric specification of this front with the parameter φ_0. Elimination of the parameter φ_0 gives the equation of the front

$$r = r(t, \varphi). \tag{2.4}$$

We convert to dimensionless variables. Let τ be the characteristic time of the invasion processes, for example, one day. We denote $r = \varepsilon\hat{r}$ and $t = \tau\hat{t}$. Then, the dimensionless functions $\hat{r}(\hat{t})$ and $\varphi(\hat{t})$ (below, the hat for the dimensionless quantities is omitted) satisfy the system

$$\begin{aligned} \frac{dr}{dt} &= B_3 r^{-1} + (B_1 + B_2 r^{-2})\sin\varphi, \qquad r(0) = 1, \\ \frac{d\varphi}{dt} &= (B_1 r^{-1} - B_2 r^{-3})\cos\varphi, \qquad \varphi(0) = \varphi_0 \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big], \end{aligned} \tag{2.5}$$

where the dimensionless parameters B_i are specified by the formulas

$$B_1 = \frac{k\tau\beta_1\delta_\gamma}{\Phi[(1 - \beta_1\varepsilon) + (1 + \beta_1\varepsilon)(R/\varepsilon)^2]},$$

$$B_2 = \frac{k\tau\beta_1\delta_\gamma R^2}{\Phi\varepsilon^2[(1 - \beta_1\varepsilon) + (1 + \beta_1\varepsilon)(R/\varepsilon)^2]}, \qquad B_3 = \frac{k\tau\delta_p}{\Phi\varepsilon^2[(\beta_1\varepsilon)^{-1} + \ln(R/\varepsilon)]}.$$

We denote $v = \sin\varphi$. Then, system (2.5) is equivalent to the following problem:

$$\begin{aligned} \frac{dr}{dt} &= B_3 r^{-1} + (B_1 + B_2 r^{-2})v, \qquad r(0) = 1, \\ \frac{dv}{dt} &= (1 - v^2)(B_1 r^{-1} - B_2 r^{-3}), \qquad v(0) = v_0 \in [-1, 1]. \end{aligned} \tag{2.6}$$

3. Qualitative Analysis of the Invasion Front. The behavior of the front is determined mainly by the parameters δ_p and δ_γ. Depending on the sign of δ_γ, three cases are possible: $\delta_\gamma = 0$, $\delta_\gamma > 0$, and $\delta_\gamma < 0$. We consider them sequentially.

3.1. Let $\delta_\gamma = 0$; then, $B_1 = B_2 = 0$ and the front is defined by the equations

$$\dot{r} = B_3/r, \qquad \dot{v} = 0, \qquad r(0) = 1, \qquad v(0) = v_0 \in [-1, 1]. \tag{3.1}$$

Since for $r = \varepsilon$,

$$H_r = -\frac{\beta_1 \delta_p}{1 + \beta_1 \varepsilon \ln (R/\varepsilon)},$$

the necessary invasion condition (1.6) leads to the inequality $\delta_p \geqslant 0$. Therefore, $B_3 \geqslant 0$ and the solution of problem (3.1) is given by the equalities

$$r(t) = \sqrt{1 + 2tB_3}, \qquad v(t) = v_0, \qquad B_3 = \frac{k\tau\delta_p}{\Phi\varepsilon^2((\beta_1\varepsilon)^{-1} + \ln (R/\varepsilon))}. \tag{3.2}$$

Hence, the front is a circumference with center at the coordinate origin and radius specified by the formula (3.2).

3.2. Let $\delta_\gamma > 0$. We first determine the conditions of satisfaction of inequalities (1.6). Using (1.4) and (2.1), this inequality is written as

$$-\delta_p - b_3 \ln (\varepsilon/R) + \max_\varphi (\sin\varphi[-b_1\varepsilon + b_2\varepsilon^{-1} - \delta_\gamma\varepsilon]) \leqslant 0,$$

i.e.,

$$\frac{\delta_p}{\varepsilon\delta_\gamma} \geqslant \frac{(1 + (R/\varepsilon)^2)(1 + (\beta_1\varepsilon)\ln (R/\varepsilon))}{(1 - \beta_1\varepsilon) + (1 + \beta_1\varepsilon)(R/\varepsilon)^2} \equiv \frac{\delta_p^*}{\varepsilon\delta_\gamma}. \tag{3.3}$$

In terms of B_i, inequality (3.3) is simplified to

$$B_3 \geqslant B_1 + B_2. \tag{3.4}$$

Inequality (3.4) is the condition on the borehole pressure p_w under which there is invasion of the mud liquids into the formation over the entire perimeter of the borehole. Below, it is assumed that inequality (3.4) is satisfied.

We analyze the invasion front using the following mathematical statement: *at each* $t > 0$, *the function* $r(t, \varphi)$ *in* (2.4)*, which specifies the front, increases monotonically with increase in* φ ($\varphi \in [-\pi/2, \pi/2]$)*. In this case,*

$$r_\varphi > 0 \quad \textit{if} \quad -\pi/2 < \varphi < \pi/2, \quad \textit{and} \quad r_\varphi = 0 \quad \textit{if} \quad \varphi = \pm\pi/2. \tag{3.5}$$

Let us prove this statement. Since $r_\varphi = r_v \cos\varphi$, it suffices to establish that

$$r_v(t, v) > 0 \qquad \text{at} \quad t > 0, \quad -1 \leqslant v \leqslant 1, \tag{3.6}$$

where

$$r = r(t, v) \tag{3.7}$$

is the equation of the boundary of the invasion front in the variables (v, r) at fixed $t > 0$.

Let us differentiate equality (3.7) with respect to v_0:

$$\frac{dr}{dv_0} = \frac{dr}{dv}\frac{dv}{dv_0}, \qquad \frac{dr}{dv} = \frac{\alpha}{\omega}, \qquad \alpha = \frac{dr}{dv_0}, \qquad \omega = \frac{dv}{dv_0}.$$

The functions $\alpha(t, v_0)$ and $\omega(t, v_0)$ satisfy the equations

$$\begin{gathered} \frac{d\alpha}{dt} = -B_3 r^{-2}\alpha - 2B_2 r^{-3} v\alpha + (B_1 + B_2 r^{-2})\omega, \qquad \alpha(0, v_0) = 0, \\ \frac{d\omega}{dt} = (1 - v^2)(-B_1 r^{-2} + 3B_2 r^{-4})\alpha - 2(B_1 r^{-1} - B_2 r^{-3}) v\omega, \qquad \omega(0, v_0) = 1, \end{gathered} \tag{3.8}$$

which are obtained by differentiation of equalities (2.6) with respect to v_0.

Let $[0, T]$ be the time interval on which the solution of the problem (2.6) is defined and let the value of r be not larger than R/ε. We show that α and ω are strictly positive in the entire interval $(0, T]$. Let $T_1(v_0)$ be the first time when $\omega(t, v_0)$ vanishes. We assume that $T_1(v_0) < T$. Considering the first equation of system (3.8) as an ordinary differential equation for α, we obtain the following representation:

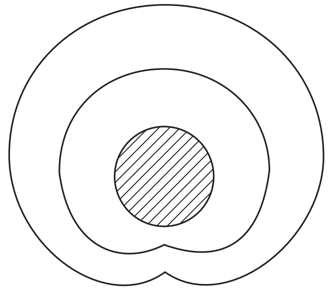


Fig. 3

$$\alpha(t, v_0) = A_1 \int_0^t \omega(s, v_0) G_1(s, v_0)\, ds, \qquad A_1(t, v_0) = \exp\Big(\int_0^t F_1(s, v_0)\, ds \Big),$$
$$F_1(t, v_0) = -B_3 r^{-2} - 2B_2 r^{-3} v, \qquad G_1(t, v_0) = (B_1 + B_2 r^{-2}) A_1^{-1}(t, v_0). \tag{3.9}$$

Therefore, $\alpha \geqslant 0$ on the interval $[0, T_1(v_0)]$. Similarly from the second equation of system (3.8), we obtain

$$\omega(t, v_0) = A_2 + A_2 \int_0^t \alpha(s, v_0) G_2(s, v_0)\, ds, \qquad A_2 = \exp\Big(\int_0^t F_2(s, v_0)\, ds \Big),$$
$$F_2(t, v_0) = -2(B_1 r^{-1} - B_2 r^{-3}) v, \qquad G_2(t, v_0) = (1 - v^2) r^{-4} A_2^{-1} (3B_2 - B_1 r^2). \tag{3.10}$$

Since

$$3B_2 - B_1 r^2 = B_1 (3(R/\varepsilon)^2 - r^2) \geqslant 2B_1 (R/\varepsilon)^2 \geqslant 0,$$

we have $\omega(T_1(v_0), v_0) > 0$, which is a contradiction. Thus, $\omega(t, v_0)$ is strictly positive on the entire interval $[0, T]$. From (3.9) and (3.10), it is easy to conclude that $\alpha(t, v_0) > 0$ on $(0, T]$. Therefore, inequality (3.6) and relations (3.5) are satisfied.

The statement proved above implies that at each fixed time, the front is the least extended in the direction $\varphi = -\pi/2$ from the center of the borehole, i.e., downward, and it is the most advanced in the direction $\varphi = \pi/2$, i.e., upward. Furthermore, the advance is larger in the direction $\varphi = \varphi_2$ than in the direction $\varphi = \varphi_1$ if the angle φ_2 is larger than the angle φ_1 (Fig. 3). The invasion depths in the directions $\varphi = \pm\pi/2$ will subsequently be defined by explicit formulas.

Let us consider the critical case

$$B_3 = B_1 + B_2. \tag{3.11}$$

Under this condition, system (2.6) for $v_0 = -1$ has a solution that does not depend on time:

$$r = 1, \qquad v = -1.$$

Hence, if the pressure difference δ_p is critical, i.e., $\delta_p = \delta_p^*$, the front "stands" at the lower point $r = 1$, $\varphi = -\pi/2$ (see Fig. 2).

Next, assuming that $B_3 > B_1 + B_2$, i.e., $\delta_p > \delta_p^*$, we find the smallest and largest offset distances from the center $r_1(t)$ and $r_2(t)$. These functions are found by solving the equations

$$\dot r_1 = \frac{B_3}{r_1} - \frac{B_2}{r_1^2} - B_1, \qquad \dot r_2 = \frac{B_3}{r_2} + \frac{B_2}{r_2^2} + B_1, \qquad r_1(0) = r_2(0) = 1. \tag{3.12}$$

Their integration leads to the formulas

$$r_1 - 1 + \ln\Big(\frac{r_1 - c_1}{1 - c_1}\Big)^n \Big(\frac{r_1 - c_2}{1 - c_2}\Big)^m = -B_1 t,$$

$$r_2 - 1 + \ln\Big(\frac{r_2 + c_2}{1 + c_2}\Big)^n \Big(\frac{r_2 + c_1}{1 + c_1}\Big)^m = B_1 t, \tag{3.13}$$

where $c_1 = b_4 + æ$, $c_2 = b_4 - æ$, $n = b_4 + (æ + b_4^2/æ)/2$, $m = b_4 - (æ + b_4^2/æ)/2$, $b_4 = B_3/(2B_1)$, and $æ^2 = B_3^2/(4B_1^2) - B_2/B_1$. Equalities (3.13) allow one to calculate the values of r_1 and r_2 at any accuracy, i.e., the mud invasion depth down and up along the vertical through the center of the borehole. From Eqs. (3.12), we can also obtain an approximate formula for the coefficient of symmetry along the vertical direction $S(t) \equiv (r_1 - 1)/(r_2 - 1)$, which characterizes the asymmetry of the front along the central vertical. Since $r_i - 1 = t\dot{r}_i(0) + o(t)$, we have

$$S(t) = \frac{r_1(t) - 1}{r_2(t) - 1} \approx \frac{1 - \xi}{1 + \xi}, \qquad \xi = \frac{B_1 + B_2}{B_3}.$$

The smaller the period of time since the beginning of invasion, the more precise this formula will be. The symmetry $S = 100\%$ if $\xi = 0$, i.e., the front is circumferential. The symmetry $S = 0$ if $\xi = 1$, i.e., the front "stands" at the lower point of the borehole.

Generally, it is possible to introduce the coefficient of symmetry along the direction φ:

$$S_\varphi(t) = \frac{r(t, -\varphi) - 1}{r(t, \varphi) - 1}, \qquad \varphi \in [0, \pi/2],$$

where $r = r(t, \varphi)$ is the equation of the front obtained from the solution $r = r(t, \varphi_0)$, $\varphi = \varphi(t, \varphi_0)$ of systems (2.5) after elimination of the parameter φ_0. Obviously, the symmetry coefficients decrease monotonically with increase in the angle and takes the smallest value for $\varphi = \pi/2$:

$$S_0(t) = 1, \quad S_{\varphi_1}(t) > S_{\varphi_2}(t) \quad \text{at} \quad 0 \leqslant \varphi_1 < \varphi_2 \leqslant \pi/2, \quad S_{\pi/2}(t) = S.$$

With time, the point of the front the nearest to the center of the borehole with the coordinates $r = r_1(t)$ and $\varphi = -\pi/2$ becomes abnormal in the sense that the neighboring left and right points of the front appear below the horizontal straight line $z = r_1(t)$ and, thus, the invasion zone is no longer a convex set. In the critical case where $B_3 = B_1 + B_2$, this property is easy to prove, and, generally, it is confirmed by fairly simple calculations.

Indeed, the horizontal straight line $z = -r_1(t)$ is specified in polar coordinates by the equation $r \sin\varphi = -r_1(t)$ or $rv = -r_1(t)$, where $v := \sin\varphi$. For this line, we have

$$\frac{dr}{dv} = \frac{r_1(t)}{v^2};$$

therefore, for the equation of the front $r = r(v, t)$, it suffices to establish that beginning from a certain time, the following inequality holds:

$$J(t) := \frac{dr}{dv}\Big|_{v=-1} > r_1(t). \tag{3.14}$$

Setting $v = -1$ in formulas (3.8), we find that the function $J(t) := (\alpha/\omega)\Big|_{v=-1}$ satisfies the equation

$$\frac{dJ}{dt} = F_3 J + B_1 + \frac{B_2}{r_1^2}, \qquad F_3(t) := \frac{4B_2}{r_1^3} - \frac{2B_1}{r_1} - \frac{B_3}{r_1^2}, \qquad J(0) = 0.$$

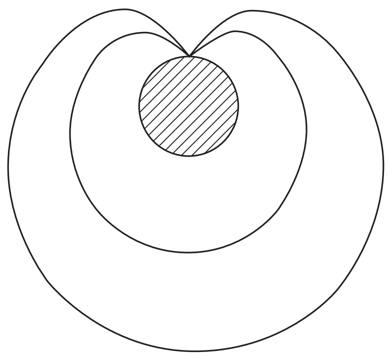
Its solution is given by the formulas

$$J = A_3(t) \int_0^t \Big(B_1 + \frac{B_2}{r_1^2(\tau)}\Big) A_3^{-1}(\tau)\, d\tau, \qquad A_3 := \exp\Big(\int_0^t F_3(\tau)\, d\tau\Big). \tag{3.15}$$

In the critical case, $r_1(t) \equiv 1$; therefore, formulas (3.15) are simplified:

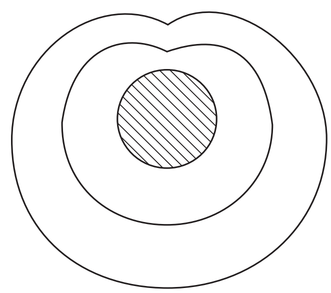
$$J(t) = (B_1 + B_2)\, e^{3(B_2 - B_1)t} \int_0^t e^{-3(B_2 - B_1)s}\, ds.$$

From this it is clear that starting from a certain time, $J(t) > 1$. Calculations of integrals (3.15) using formulas (3.13) show that inequality (3.14) is generally satisfied for rather large t. Thus, with time, the point of the invasion front directly beneath the center of the borehole inevitably becomes abnormal.



$\rho_w > \rho_f, \delta_p = \delta_p^*$

Fig. 4



$\rho_w > \rho_f, \delta_p > \delta_p^*$

Fig. 5

3.3. Let $\delta_\gamma < 0$. In this case, mathematical analysis of the equations of the front is performed by the same scheme as for $\delta_\gamma > 0$. For this, it suffices to replace the positive parameters B_1 and B_2 by the negative parameters: $B_1 := -|B_1|$ and $B_2 := -|B_2|$. We give only the final results. The necessary condition of invasion (1.6) is satisfied if the following inequality holds:

$$B_3 \geqslant |B_1| + |B_2|.$$

At each fixed time, the monotonicity property of the front is expressed by the conditions

$$r_\varphi < 0 \quad \text{if} \quad -\pi/2 < \varphi < \pi/2, \quad \text{and} \quad r_\varphi = 0 \quad \text{if} \quad \varphi = \pm\pi/2. \tag{3.16}$$

For $B_3 = |B_1| + |B_2|$, the front "stands" at the top of the borehole (Fig. 4). Generally, the front is a symmetric reflection (relative to the horizontal axis $x_2 = 0$) of the front obtained for the case $\delta_\gamma := |\delta_\gamma|$ (Fig. 5). This follows from the fact that system (2.5) has the following symmetry. Let $B_1 < 0$, $B_2 < 0$, and $B_3 > 0$. We denote the solution of problem (2.5) by $r(t; B_1, B_2, B_3, \varphi_0)$, $\varphi(t; B_1, B_2, B_3, \varphi_0)$ and introduce the functions $r_1(t) = r(t; B_1, B_2, B_3, \varphi_0)$ and $\varphi_1(t) = -\varphi(t; B_1, B_2, B_3, \varphi_0)$. Then, it is easy to verify that (r_1, φ_1) is a solution of problem (2.5) if B_1 is replaced by $|B_1|$, B_2 by $|B_2|$, B_3 by B_3, and φ_0 by $-\varphi_0$, i.e.,

$$\frac{dr_1}{dt} = B_3 r_1^{-1} + (|B_1| + |B_2| r_1^{-2}) \sin\varphi_1, \qquad r_1(0) = 1,$$

$$\frac{d\varphi_1}{dt} = (|B_1| r_1^{-1} - |B_2| r_1^{-3}) \cos\varphi_1, \qquad \varphi_1(0) = -\varphi_0 \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big].$$

For $\delta_\gamma < 0$, the symmetry coefficient is determined similarly:

$$S_\varphi(t) = \frac{r(t,\varphi) - 1}{r(t,-\varphi) - 1}, \qquad \varphi \in \Big[0, \frac{\pi}{2}\Big].$$

Here $r = r(t, \varphi)$ is the equation of the front.

4. Numerical Analysis of the Invasion Front. The monotonicity property of the front along the angular variable φ, expressed by conditions (3.5) or (3.16), gives only a qualitative picture of the invaded zone for fixed times. To study the geometry of the front and its dynamics, we performed calculations of system (2.5) using the Runge–Kutta method. The front was plotted as a set of points $(r_m^n, \varphi_m^n) = (r(t_n, \varphi_m), \varphi(t_n, \varphi_m))$ for discrete times t_n and for a discrete set of initial data φ_m, where $r(t_n, \varphi_m)$, $\varphi(t_n, \varphi_m)$ is a numerical solution of the Cauchy problem (2.5).

The calculations were performed for a certain typical set of physical parameters used in analysis of straight holes [1]: borehole radius $\varepsilon = 0.108$ m, characteristic dimension of the formation $R = 10$ m, porosity $\Phi = 0.2$, and reduced filtration coefficient $k_0 \equiv k\gamma_f = 0.1$ m/day.

In determining the numerical value of the resistance coefficient β_1, we assume that it depends insignificantly on the difference of the weights δ_γ and, hence, can be calculated by formulas (3.2) for experiments in which $\delta_\gamma = 0$. For a certain control pressure difference δ_p^c, let the invasion zone reached a value d (dimensionless value d/ε) for the control time τ (corresponding to the dimensionless time $t = 1$). Then,

$$\frac{d}{\varepsilon} = \sqrt{1+2B_3}, \qquad B_3 = \frac{k\tau\delta_p^c}{\Phi\varepsilon^2((\beta_1\varepsilon)^{-1} + \ln(R/\varepsilon))}.$$

From this, we obtain the reduced resistance coefficient $\beta_0 = \beta_1 k\gamma_f$:

$$\beta_0 = \frac{\Phi k_0(d^2 - \varepsilon^2)}{2\varepsilon k_0\tau\delta_p^{c,0} - \varepsilon\Phi(d^2-\varepsilon^2)\ln(R/\varepsilon)}, \qquad \delta_p^{c,0} = \frac{\delta_p^c}{\gamma_f}. \tag{4.1}$$

As the control parameters, we use those calculated in [1] for a straight hole with a radially symmetric invasion front: $\tau = 1$ days, $\delta_p^{c,0} = 100$ m, and $d = 0.508$ m. Then, from (4.1) we find that $\beta_0 = 2.3 \cdot 10^{-3}$ 1/day.

The dimensionless parameters B_i can be written as

$$B_1 = \frac{\tau\beta_0(1-\gamma_w/\gamma_f)}{\Phi(1-\beta_0\varepsilon/k_0 + (1+\beta_0\varepsilon/k_0)(R/\varepsilon)^2)},$$

$$B_2 = B_1\Big(\frac{R}{\varepsilon}\Big)^2, \qquad B_3 = \frac{\tau k_0\delta_p^0}{\Phi\varepsilon^2(k_0/(\varepsilon\beta_0) + \ln(R/\varepsilon))};$$

their numerical values can be found from these formulas.

The ratio of the weights γ_w/γ_f varies from 0.95 to 1.20. The reduced pressure jump $\delta_p^0 \equiv \delta_p/\gamma_f$ can reach 600 m.

The calculations show that the invasion front is extended with time along any half-line $\varphi = \text{const}$. Figures 2–5 illustrate this for relatively small values of δ_p^0 for two successive times $t_1 = 0.5$ day and $t_2 = 1$ day when $\varepsilon = 0.108$ m, $R = 10$ m, $\Phi = 0.2$, $k_0 = 0.1$ m/day, $\tau = 1$ day, $\beta_0 = 2.3 \cdot 10^{-3}$ 1/day, and $\gamma_w/\gamma_f = 0.95$ or $\gamma_w/\gamma_f = 1.20$.

In conclusion, we give results of calculation of the vertical symmetry S for five successive times (days), when $\gamma_w/\gamma_f - 1 = 0.20$, and $\delta_p^0 = 100$ m: 0.9998 for 0.5 day, 0.99991 for 1 day, 0.99994 for 2 days, 0.99997 for 4 days, and 0.99998 for 5 days. These data were obtained by numerical solution of the Cauchy problem (2.5).

We calculate S by the approximate formula $S \simeq (1-\xi)/(1+\xi)$. Since

$$\xi \equiv \frac{|B_1| + |B_2|}{B_3} = \frac{\gamma_w/\gamma_f - 1}{\delta_p^0}\,\frac{(\varepsilon^2 + R^2)(1/\varepsilon + (\beta_0/k_0)\ln(R/\varepsilon))}{1 - \varepsilon\beta_0/k_0 + R^2/\varepsilon^2 + \varepsilon\beta_0 R^2/(k_0\varepsilon^2)},$$

we have $\xi = 2.2 \cdot 10^{-4}$ and $S = (1-\xi)/(1+\xi) = 99.956\%$. Thus, under typical drilling conditions, the invasion front is nearly circumferential.

At the drilling stage, the large pressure difference ensures mud circulation. After termination of drilling, the pressure difference sharply reduces and the second stage of invasion occurs, in which the invading mud fluid is "underlain" by the next portion of the mud, which is now at lower pressure. Thus, the problem of deformation of a nearly circumferential front produced at the drilling stage arises. In view of the aforesaid, this work can be regarded as development of a procedure for calculating the initial conditions for the second stage of invasion.

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